

# BPS Wall Crossing and Topological Strings

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By embedding  $N = 2$  gauge theories in string theory and utilizing string dualities we map the counting of BPS states with arbitrary electric and magnetic charges to computations of an A-model topological string on an associated geometry constructed from the data of the SW curve. We show how the conjecture of Kontsevich and Soibelman regarding wall crossing, as well as a more refined version which captures the spin content of BPS states, is a natural consequence. Chern-Simons theory realized on A-branes and a twistorial construction play key roles.

## 1. Introduction

The study of BPS states has been a key ingredient in understanding quantum aspects of supersymmetric theories. In particular one can obtain exact information about their masses and degeneracies as has been known for a long time [1]. It is thus not surprising that BPS objects have played a key role in developing string dualities.

One of the important facts about BPS states, which was discovered also a long time ago [2], in the context of supersymmetric  $(2, 2)$  theories in 2 dimensions, is that their degeneracy can jump when the central charges are given by points on the complex plane. When the phases of two BPS particles interchange, then they can disappear (or new ones may appear). At the face of it this sounds contradictory with continuity of correlation functions in path-integrals. However it was shown in [3] how this contradiction is resolved: When a BPS particle decays after crossing a wall what used to be a single BPS particle contribution to correlation functions gets replaced by multi-particle contributions. Moreover the continuity of a specific path-integral computation, captured by the index

$$\text{Tr}(-1)^F \exp(-\beta H)$$

can be used to give a general recipe as to how the degeneracy of BPS particles jump as we cross the wall.

In the context of  $N = 2$  supersymmetric theories in four dimensions, the central charges of electric and magnetic particles are also given by points on the complex plane, and thus there can be jumps. In fact it was observed by Seiberg and Witten [4] that not only there can be jumps, but that there has to be jumps for the theory to be consistent. In particular they obtained a very intricate pattern of BPS jumps which are far more complicated than the ones encountered in the 2d theories. Namely in such cases sometimes infinitely many BPS particles disappear as one crosses the wall. That this could happen follows because the BPS central charges lie on the projection of an integral lattice to the complex plane and thus when any two get aligned, infinitely many central charges get aligned due to linearity. However there was no systematic way to compute these degeneracies and their jumps.

In the context of realizing  $N = 2$  gauge theories in 4 dimensions, with the advent of string dualities, the situation became a bit more clear: It was realized in [5] that by geometric engineering and an application of a series of dualities,  $N = 2$  gauge systems are mapped in the type IIA setup to NS 5-branes whose worldvolume is  $\mathbf{R}^4 \times \Sigma$  where  $\Sigma$  is the

Seiberg-Witten curve. Moreover the BPS states were mapped to D2 branes ending on the 5-brane. Using this, it became possible to compute the BPS degeneracies directly, at least in some simple situations, and see that indeed they jump according to what was expected [5][6]. For certain situations, a wall-crossing formula was proposed in [7]. However it was not possible to extend this to the general case.

Recently a surprising conjecture was put forward by Kontsevich and Soibelman [8] as to how the BPS degeneracies jump. It was shown in [9] how this result can be derived by demanding that the contribution of BPS particles to hyperkähler metrics obtained upon compactification on a circle, be continuous. Again, as in the 2d case, the continuity of the correlation functions was due to multi-particle contributions replacing single particle contributions.

In this paper we offer an alternative derivation of the KS conjecture for all the  $N = 2$  theories that can be associated to a Seiberg-Witten curve.<sup>1</sup> In fact we go further: We show that the computation of BPS degeneracies can be mapped to a computation of open topological A-model strings, for which a great deal is known. More precisely, we compactify this IIA theory on a circle and use an 11/9 flip which changes which direction corresponds to the circle of M-theory. In other words we lift the 4d type IIA theory to M-theory, in which case the NS 5-brane gets mapped to M5 brane, and compactify the theory on a circle, which we now view as the 11-th circle. In this way we get back to IIA theory, but now the NS 5 brane is replaced by a D4 brane whose worldvolume is  $\mathbf{R}^3 \times \Sigma$ . In this context the BPS particles correspond to  $D2$  branes ending on  $D4$  branes. This is close to the situations where we know how to count their degeneracy [11]: They correspond to partition functions of topological A-model strings where D4 brane wraps over a Lagrangian cycle of Calabi-Yau threefold. There is one difference however: We are in a situation corresponding to twice as much supersymmetry. To remedy this, motivated by the twistor construction, we compactify on a circle down to 2 dimensions, where the Seiberg-Witten curve fibers in an interesting way over the circle. In this way we can break half of its supersymmetry. We are then down to 2 dimensions, where D4 brane wraps  $\Sigma \times S^1 \times \mathbf{R}^2$  and topological string computes specific F-term corrections to the  $(2, 2)$  supersymmetric theory in 2 dimensions as in [11]. Moreover these get mapped to computation of correlation functions of a  $U(1)$

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<sup>1</sup> There are some solvable  $N = 2$  systems, such as quiver theories based on E-groups [10] which suggest that the corresponding  $N = 2$  geometry is not given in general by a Seiberg-Witten curve, but rather by a local Calabi-Yau threefold.

Chern-Simons theory living on  $\Sigma \times S^1$ . The BPS particles ending on the D4 brane induce Wilson Loop operators on the Chern-Simons theory. Moreover we choose the fibration structure of  $\Sigma$  over  $S^1$  such that each BPS particle with a given phase for the central charge attaches to the D4 brane at the corresponding angle on the circle! Viewing the circle as the time direction, leads to the computation of the path-integral of Chern-Simons theory by taking the trace of these operator in a time ordered fashion; and since time and phase are identified here, it means ordering in terms of the phase of their BPS charge. The fact that partition function of the A-model topological strings can change only in a continuous way as we change parameters, leads directly to the results of KS <sup>2</sup>. One way to phrase our results is the following: The BPS degeneracies lead to a particular (infinite) combination of link invariants on  $\Sigma \times S^1$ . As we cross the walls of marginal stability the links change topological type and thus the associated Chern-Simons invariants change. However, the BPS degeneracies which dictate which combinations of the invariants we take also change in such a way that the full amplitude does not change!

We get two further refinements: First of all, it is known that the spin of the BPS particles are also encoded in the topological string in terms of the coupling constant of topological string. Using this we get a refinement of KS conjecture, which was already suggested in [13]. Moreover we get a further refinement, as in [11][14] by considering more than one D4 brane. Namely there is a further refined integrality which is encoded in topological string partition function having to do with the fact that a D2 brane can end on multitude of boundary circles on  $\Sigma$ , in terms of Wilson loop correlations of a  $U(N)$  Chern-Simons theory.

The organization of this paper is as follows: In section 2 we review aspects of topological A-model string. In particular we discuss integrality properties of open and closed string amplitudes as well as the relation to Chern-Simons theory and Wilson loop operators. In section 3 we discuss 5-brane geometries which realize  $N = 2$  systems in 4d as well as their further compactifications. We discuss a general class of twistorial compactifications which is relevant for our application. In section 4 we use this to derive the KS conjecture and its refinement. In section 5 we attempt to explicitly construct such branes, where we point out that the most obvious construction, even though does not lead to Lagrangian branes, is sufficiently close to one, to establish our result. Finally in section 6 we end with some concluding thoughts.

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<sup>2</sup> The idea to use topological strings as a tool for gaining insight into wall crossing phenomena has been considered in a different context in [12].

## 2. Aspects of A-model Topological Strings

Topological A/B strings compute F-terms for type IIA/B compactifications on Calabi-Yau threefold. In this paper we would be mainly interested in type IIA case, for reasons that will become clear. Here we review some aspects of A-model topological strings that we need.

A-model topological string [15] ‘counts’ holomorphic maps from string worldsheet to Calabi-Yau threefolds. It is known that due to the fact that generically these maps come with moduli, the counting in general would lead to rational numbers. However it was shown in [16] that there is an integrality in this expansion given by

$$Z = \exp \left[ \sum_{Q,j} N_{Q,j} \sum_n \frac{1}{n} e^{-nt \cdot Q} (q^{n/2} - q^{-n/2})^{2j-2} \right]$$

where  $q = \exp(-g_s)$ ,  $t$  is the Kahler moduli and  $N_{Q,j}$  are integers that count the number of BPS branes in one higher dimension. More precisely, consider M-theory on the same Calabi-Yau in one higher dimension. Then  $N_{Q,j}$  counts the net number of BPS  $M2$  branes in the class  $Q \in H_2(CY, \mathbf{Z})$  and  $SU(2)_L$  spin  $j$ , where  $SU(2)_L \times SU(2)_R$  is the rotation group in 5 dimensions. This derivation used the duality of IIA theory with M-theory, where the  $M2$  branes contribute to F-terms of type IIA theory on Calabi-Yau generated by BPS particles going around the M-theory circle.

One can also consider open topological strings. In that case one introduces D-branes wrapping Lagrangian submanifolds [17] and considers holomorphic maps from the worldsheet with boundary, where the boundary lies on the D-brane. Moreover it was shown in [17] that on the topological D-brane lives a Chern-Simons theory. Moreover for every holomorphic curve of area  $A$  ending on a loop  $\gamma$  in the D-brane, one gets an insertion of the Wilson loop in the Chern-Simons Lagrangian, which schematically looks as  $U_\gamma \exp(-A)$ , where  $U_\gamma$  is the holonomy of the gauge field around  $\gamma$ . There are also integrality properties for these corrections to Chern-Simons theory which have a similar structure to that of the closed string sector [11]. For simplicity let us consider the case where we have one D-brane wrapping a Lagrangian cycle with one non-trivial 1 cycle. The generalization to when we have more than 1-cycle is straight-forward. Also let us assume that there are no non-trivial 2-cycles in the CY (as will be the main focus of this paper). The general case is discussed in [11]. Let  $U$  denote the holonomy of the  $U(1)$  gauge field around that cycle. Then the open topological string partition function is given by

$$Z = \exp \left[ \sum_{j,m} N_{j,m} \sum_n \frac{U^{nm}}{n} \frac{q^{nj}}{(q^{n/2} - q^{-n/2})} \right] \quad (2.1)$$

where  $N_{j,m}$  are some integers counting certain BPS states in the dual M-theory. More precisely, consider type IIA string with D4 branes wrapping the Lagrangian cycle of CY and filling a 2d subspace of spacetime. This lifts in M-theory to an M5 brane wrapping the Lagrangian cycle and filling  $\mathbf{R}^3$ .  $N_{j,m}$  counts the net number of M2 branes ending on M5 brane, whose boundary wraps the non-trivial cycle of the Lagrangian brane  $m$  times and has spin  $j$  in  $\mathbf{R}^3$ . Note that if we have more than one-cycle in the Lagrangian submanifold we get similar contributions for each cycle and for each M2 brane ending on the M5 brane. Furthermore this counting can be refined if we consider  $K$  instead of 1 brane wrapping the Lagrangian brane. This integrality can also be expressed in similar terms [11][14]. For more than one Lagrangian brane, the label  $m$  gets replaced by the representation label  $\mathcal{R}$  and the  $(U^m)^n$  get replaced by  $\text{tr}_{\mathcal{R}} U^n$ .

If we have more than one cycle in the Lagrangian submanifold, as will be the case for applications we will be interested, we get operator insertions in the CS theory for M2 branes, ending in a class  $\gamma$ , and of spin  $j$  given by

$$O_{j,\gamma} = \exp \left[ N_{j,\gamma} \sum_n \frac{U_\gamma^n}{n} \frac{q^{nj}}{(q^{n/2} - q^{-n/2})} \right] \quad (2.2)$$

Note that these are insertions in the CS path integral, and so we need to compute the partition functions of the CS theory with these insertions to get the full amplitude for A-model topological strings in the presence of Lagrangian A-branes.

### 3. Engineering $N = 2$ Theories in 4d and Their Compactifications

In this section we consider the chain of dualities which map the 4d  $N = 2$  theory to an associated 2d theory with  $(2, 2)$  supersymmetry. In the next section we will use this setup to show how the BPS degeneracies can be recovered from open A-model topological strings. As a byproduct, this leads to a simple derivation of a refined version of KS conjecture.

We will start with the theory in 4d realized by an M5 brane wrapping  $\Sigma \times \mathbf{R}^4$  where  $\Sigma$  is a curve defined by <sup>3</sup>

$$F(z_1, z_2) = 0, \quad (x, z_3) = 0.$$

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<sup>3</sup> Note that these classes of 5-branes span a large class of known  $N = 2$  theories: In particular it is known that all the  $N = 2$  theories geometrically engineered in type IIA using 3-folds are of this type [10], where by mirror symmetry we get a type IIB theory on the geometry

$$uv + F(z_1, z_2) = 0,$$

where  $x$  is a real line,  $z_3$  is a complex plane, and for simplicity we assume  $(z_1, z_2) \in \mathbf{C}^2$ .

In this setup the BPS states correspond to M2 branes ending on  $\Sigma$ . Let  $D$  denote such an M2 brane with  $\partial D \subset \Sigma$ . The central charge of the  $N = 2$  theory for the M2 brane is given by

$$Z = \int_D i dz_1 \wedge dz_2$$

and its mass is given by

$$|Z| = \int_D i e^{-i\theta} dz_1 \wedge dz_2$$

In particular  $e^{i\theta}$  measures the phase of  $i dz_1 \wedge dz_2$  restricted to the M2 brane. Note in particular that fixing the  $\theta$  for which there is a BPS particle picks out a given ray in  $H_1(\Sigma)$ , which corresponds to that given by  $\partial D \in H_1$ . This implies that on the boundary  $\partial D$  the one form  $\lambda = z_2 dz_1$ , which is identified with the Seiberg-Witten differential has a definite phase. Viewing  $z_2(z_1) = W'(z_1)$  maps this boundary curve on  $\Sigma$  to a straight line in the  $W$  plane, just as is the case in the  $N = 2$  LG theory in 2d [2]. This fact was used to capture BPS degeneracies in some simple cases [5][6], and confirm some of the expected BPS jumping phenomena as one crosses the walls of marginal stability. However, in order to find a more systematic way to capture the BPS degeneracies we need some new ideas, which we will now turn to.

Consider compactifying this theory on a circle, down to 3 dimensions. However, now we view the 4th dimension as the M-theory direction. This leads to a type IIA theory where we have a D4 brane wrapping  $\Sigma$  and filling the  $\mathbf{R}^3$  spacetime. Next we consider further compactification on a circle:

$$\mathbf{R}^3 \rightarrow S^1 \times \mathbf{R}^2.$$

We thus end up with a theory living on the D4 brane in 2 dimensions with  $(4, 4)$  supersymmetry. To see the structure of this theory better, let us combine this  $S^1$  with the  $x$ -line, and call it a new cylindrical coordinate  $\zeta = \exp(i\theta - x)$ . We can view this new geometry as a non-compact flat Calabi-Yau, given by

$$(z_1, z_2, \zeta)$$

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which in turn using T-duality [18], gets mapped to NS 5-brane on  $F(z_1, z_2) = 0$  [5]. This can be lifted to M-theory where the NS5 brane becomes the M5-brane wrapping this curve. Alternatively we can also consider brane constructions of  $N = 2$  theories as in [19] which leads directly to such geometries for constructing  $N=2$  theories (see also the recent work [20]).

with the rest of the spacetime being

$$(z_3, \mathbf{R}^2)$$

where D4 brane wraps the  $\mathbf{R}^2$  (at  $z_3 = 0$ ) and a 3d internal submanifold

$$F(z_1, z_2) = 0, \quad |\zeta| = 1.$$

Since  $\mathbf{C}^2$  is hyperkähler, by a rotation of complex structure we can view this 3d submanifold as a Lagrangian submanifold of the CY. On the non-compact worldvolume of D4 brane we get a  $(4, 4)$  supersymmetric theory in 2 dimensions.

We now consider the topological A-model in the internal space. This would have worldsheet instantons which wrap holomorphic curves in  $\mathbf{C}^2$  which end on  $\Sigma \times S^1$ . All such curves are at a fixed value of  $\theta \in S^1$ . These are precisely the BPS particles we are after! More precisely, as in [11] we know that the degeneracy and the spin of M2 branes ending on M5 branes are captured by topological strings, as reviewed in the last section. However, the story is not so simple: Here we have twice as much supersymmetry compared to the one discussed in [11] because the space is flat (or more generally hyperkähler times a flat space). This is reflected by the fact that when we talk about holomorphic curves we have a family of possible choices for that notion, and therefore the amplitudes vanish due to higher supersymmetry. Namely for every holomorphic curve, we have a  $U(1)$  rotation symmetry of the phase of  $\zeta$  given by where they intersect the  $S^1$ . We thus get a family of such curves with no fixed point, and thus the fermion zero mode along that direction kills the amplitude<sup>4</sup>. For this reason we modify our geometry so that we have half as much supersymmetry, i.e.  $(2, 2)$  supersymmetry in 2 dimensions.

The basic idea is the following: The BPS particles correspond to holomorphic M2 branes with respect to different complex structure. In particular for a fixed  $\theta$ , consider a new pair of complex variables given by

$$w_1 = z_1 + e^{i\theta} \bar{z}_2 \quad w_2 = e^{-i\theta} z_2 - \bar{z}_1$$

We can view this as a special case of the twistorial construction where

$$w_1 = z_1 + \zeta \bar{z}_2 \quad w_2 = \zeta^{-1} z_2 - \bar{z}_1, \tag{3.1}$$

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<sup>4</sup> Another way to see this is that when we change the complex structure, the holomorphic curve ceases to be holomorphic and we have nothing contributing to the amplitudes.



Consider a new Kähler form given on  $\mathbf{C}^2$  by

$$-ik_\theta = dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2$$

Then it is easy to see that

$$-i k_\theta = 2 dz_1 \wedge dz_2 \cdot e^{-i\theta} - 2 d\bar{z}_1 \wedge d\bar{z}_2 \cdot e^{i\theta}$$

and

$$dw_1 \wedge dw_2 + d\bar{w}_1 \wedge d\bar{w}_2 = 2 dz_1 \wedge dz_2 \cdot e^{-i\theta} + 2 d\bar{z}_1 \wedge d\bar{z}_2 \cdot e^{i\theta}.$$

From the first identity, it is clear that  $F(z_1, z_2) = 0$  is a Lagrangian subspace of  $\mathbf{C}^2$  with respect to this Kähler form. Moreover, from the second one we see that holomorphic M2 branes ending on  $F = 0$ , will be such that

$$\text{Im}(i dz_1 \wedge dz_2 \cdot e^{-i\theta})|_{M_2} = 0$$

$$0 < k_\theta|_{M_2} = 2 \text{Re}(i dz_1 \wedge dz_2 \cdot e^{-i\theta})|_{M_2} = 2 i dz_1 \wedge dz_2 \cdot e^{-i\theta}|_{M_2}$$

so that the restriction of  $i dz_1 \wedge dz_2 e^{-i\theta}|_{M_2}$  will be real positive. In other words, BPS particles whose central charge have phase  $\theta$  are holomorphic M2 branes with this choice of complex structure.

This construction suggests what we need to do: We should change the geometry so that the internal geometry of the brane is given by an A-brane, with topology of  $\Sigma \times S^1$  and where for each  $\theta$  the restriction of the Kähler form of the 3-fold to  $\mathbf{C}^2$  gives the structure given above. In other words the geometry should be fibered in a twistorial fashion, as discussed above. In this way the BPS particles will correspond to holomorphic M2 branes which attach to  $S^1$  at specific points given by the phase of their central charge. The exact way this construction is done would not affect our conclusions, which will be presented in the next section. We will discuss one such construction in section 5. We find that in this simplest construction, the  $\Sigma \times S^1$  brane though not quite Lagrangian, behaves as one, since every 2 dimensional subspace of it (given by  $\zeta = \text{const.}$ ) is special Lagrangian. The A-model amplitudes are well defined, because the path-integral is localized on horizontal slices (w.r.t.  $\zeta$ ) for which the Lagrangian condition is satisfied.

## 4. Derivation Of BPS Degeneracies And Their Jumps

In this section, using the ideas of the previous section, we show how the topological string captures the BPS degeneracies. Furthermore we show an immediate consequence of it is a refined version of the Kontsevich-Soibelman conjecture [8].

In the previous section we have shown that a particular twistorial compactification of  $N = 2$  theory from 4 to 2 dimensions can yield a theory with  $(2, 2)$  supersymmetry. Namely we found that, by a chain of dualities, this is equivalent to a theory living on the D4 brane wrapping a submanifold  $\Sigma \times S^1$  and filling the  $\mathbf{R}^2$ . Topological A-model string computes F-terms for this  $(2, 2)$  theory as reviewed in section 2. As discussed there, the partition function of the open topological string receives corrections from holomorphic curves ending on  $\Sigma \times S^1$ . As noted in the previous section the curve projects to a particular point on  $S^1$  and moreover, these are in turn equivalent to M2 branes ending on the M5 brane. Thus the partition function of open topological strings living on the D4 brane has the full information about the quantum numbers, the degeneracy and the spin of all BPS particles M2 branes ending on M5 branes. In particular the corrections to the Chern-Simons theory living on  $\Sigma \times S^1$  is given by

$$Z_{CS} = \langle \prod_{\gamma, j} O_{j, \gamma}(\theta_\gamma) \rangle$$

where

$$O_{j, \gamma} = \exp \left[ N_{j, \gamma} \sum_n \frac{U_\gamma^n}{n} \frac{q^{nj}}{(q^{n/2} - q^{-n/2})} \right]$$

and  $N_{j, \gamma}$  denotes the net number of BPS states with spin  $j$  and electric/magnetic charge given by  $\gamma$  and  $U_\gamma$  denotes the Wilson loop operator along  $\gamma$ . The parameter  $q$  is related to the coupling of the topological string which is in turn the same as the coupling of the CS theory:  $q = \exp(-g_s)$ . Note that these Wilson loops which arise due to holomorphic curves, are localized on specific location  $\theta_\gamma$  on the  $S^1$ , due to the fact that BPS states with that given  $\gamma$  can be holomorphic only for that angle as discussed in the previous section. We thus see that the partition function of the topological A-model for this CY captures the electromagnetic BPS states of the 4d  $N = 2$  theory including the information about their spin! Note that we can insert arbitrary extra observables (such as by introducing additional branes, etc.) into this topological theory, and we can thus in principle extract all the  $N_{\gamma, j}$  using topological A-model.

As an application of this construction we are ready to derive the KS conjecture: For that we view the Chern-Simons theory on  $\Sigma \times S^1$  in the operator formulation. Namely

we view  $S^1$  as time and  $\Sigma$  as space. Then, as is well known [21] the classical phase space is given by the space of flat  $U(1)$  connections on  $\Sigma$ . This is naturally identified with the Jacobian of  $\Sigma$  with the natural symplectic structure giving the Poisson bracket, which arises from the  $AdA$  term in the CS Lagrangian. If we parameterize the holonomies of the  $U(1)$  along canonical basis of 1-cycles on  $\Sigma$  given by  $(A_i, B_j)$ , by the angles  $(\eta_i, \phi_j)$ , we have classically

$$\{\eta_i, \phi_j\} = \delta_{ij}$$

In the quantum theory this leads to the commutation relation

$$[\eta_i, \phi_j] = ig_s \delta_{ij}$$

In this basis the holonomy of the operator  $U_\gamma$ , for  $\gamma = n^i A_i + m^j B_j$  is given by

$$U_\gamma = \exp[i(\eta_i n^i + \phi_j m^j)]$$

Moreover note that in the operator formulation we need to time order the fields. Since the time is identified with the phase of the BPS states, this simply means  $\theta$  ordering of the operators  $O_{j,\gamma}$ . Note that since  $[U_{n\gamma}, U_{m\gamma}] = 0$  there is no ambiguity when both of these operators appear at the same time  $\theta$ . We thus find that the time/ $\theta$  ordered operator

$$T(\prod_{\gamma,j} O_{j,\gamma}(\theta_\gamma))$$

is the evolution operator. Now consider changing the parameters of the  $N = 2$  theory in 4 dimensions. The BPS states may get aligned, and cross walls of marginal stability. If this happens the operator ordering of their contributions to the Chern-Simons theory gets changed. However the path-integral of the A-model, and thus the evolution operator given above is continuous! Thus the contributions of the BPS states to the CS theory Wilson loops just after crossing the wall of marginal stability should be the same as the one just before crossing. As we cross the wall, some phases of the central charges of some BPS states get aligned. Let us focus on two such charges, given by the classes  $\gamma, \gamma' \in H_1(\Sigma, \mathbf{Z})$ . This means that for the same  $\theta$  there are contributions to the topological strings involving  $U_\gamma$  and  $U_{\gamma'}$ . If  $\langle \gamma, \gamma' \rangle \neq 0$  then the Wilson loops pass through the same point(s) in  $\Sigma$ . This means that as a cycle in  $\Sigma \times S^1$  the linking of these Wilson loop observables will change as we cross the wall. Since the BPS degeneracies dictate which combinations of them we obtain in the A-model path-integral (according to dilog and its generalization) we need

to change these degeneracies, in order for the combinations of CS link amplitudes not to change.

By inserting extra observables in the CS theory, we can localize this statement along the twistor circle. The  $g_s \rightarrow 0$  limit of this statement (i.e. the disc amplitudes of the A-model) leads to a Poisson action on the phase space which reproduces the symplectic morphism of the KS. Here we have uncovered the reason for the appearance of dilogs in their construction. Moreover, we have a refinement of their statement; namely we can consider this continuity of the operator for arbitrary  $g_s$ . This captures the spin dependence of the BPS states and leads to a refinement of KS proposed in [13].

## 5. Construction of the A-brane

Consider the geometry we ended up in section 3, namely the complex space given by  $M = \mathbf{C}^2 \times \mathbf{C}^*$  where  $\mathbf{C}^2$  is parameterized by  $w_1, w_2$  and  $\mathbf{C}^*$  by  $\zeta$ . These are related to the  $z_i$  by

$$w_1 = z_1 + \zeta \bar{z}_2 \quad w_2 = \zeta^{-1} z_2 - \bar{z}_1$$

In this geometry we consider the 3-dimensional submanifold given by  $K = \Sigma \times S^1$  defined by

$$F(z_1, z_2) = 0, \quad |\zeta| = 1.$$

Let  $C \subset \mathbf{C}^2 \times \mathbf{C}^*$  be a holomorphic curve with boundary in  $K$ . The projection of  $C$  on the second factor,  $\mathbf{C}^*$ , is necessarily a point on the unit circle.<sup>5</sup> All such curves  $C$  will end on the  $F = 0$  curve which is special Lagrangian inside  $\mathbf{C}^2$ , and each one will map to a point on the  $\zeta = e^{i\theta}$  cylinder, with  $\theta$  corresponding to the phase of the corresponding BPS states.

One should ask if the  $K$  we just constructed is Lagrangian, as is generically required for A-branes. In order to do this we have to pick a Kähler form. We equip  $M$  with the natural flat Kähler form

$$-i\omega_M = dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2 + \frac{d\zeta}{\zeta} \wedge \frac{d\bar{\zeta}}{\bar{\zeta}},$$

as well as the holomorphic 3-form

$$\Omega_3 = \frac{d\zeta}{\zeta} \wedge dw_1 \wedge dw_2,$$

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<sup>5</sup> This is because the  $S^1$  is a non-contractible cycle in the geometry.

making  $M$  into a (flat) Calabi-Yau 3-fold.

In  $M$ , we consider the real codimension 1 submanifold

$$R = \{ \zeta = e^{i\theta}, \quad 0 \leq \theta < 2\pi \},$$

that is the twistor space  $M$  restricted to the equator of the twistor sphere. We write  $\varpi$  for the symplectic 2-form  $-i\omega_M|_R$ . We notice the following identity

$$\varpi = 2 dz_1 \wedge d(e^{-i\theta} z_2) - 2 d\bar{z}_1 \wedge d(e^{i\theta} \bar{z}_2). \quad (5.1)$$

This identity implies that for fixed  $\theta$ ,

$$\varpi_\theta|_K = 0,$$

as we had noted in section 3. However we now see that  $\omega_M|_K \neq 0$ , due to the  $z_2 dz_1 d\theta e^{-i\theta}$  term, if we consider the non-horizonatal directions, i.e., directions for which  $\theta \neq \text{const}$ . In other words, if there were configurations in the bulk of the closed worldsheet which had non-constant  $\zeta$ , then this would have led to a problem in defining the string amplitudes in the presence of  $K$ . However since all the holomorphic maps are automatically horizontal, and  $K$  is special Lagrangian in those directions, we have no difficulty in defining the path-integral measure for the A-model.<sup>6</sup> One way to understand this is to consider rescaling the Kähler metric in the  $\zeta$  direction by a large number  $t$ :

$$\omega_t = i t \frac{d\zeta}{\zeta} \wedge \frac{d\bar{\zeta}}{\bar{\zeta}} + i dw_1(\zeta) \wedge d\bar{w}_1(\bar{\zeta}) + i dw_2(\zeta) \wedge d\bar{w}_2(\bar{\zeta}),$$

and evaluate the path-integral for the corresponding  $\sigma$ -model, with Dirichlet boundary conditions on  $K$ , in the limit  $t \rightarrow \infty$ .

We perform first the integral over the non-zero modes of the field  $\zeta$ , for a fixed configuration of the fields  $z_1, z_2$ , and boundary condition on  $\zeta$  given by  $|\zeta|_{\partial D} = 1$ . The integral is Gaussian, and the result is an effective action for the fields  $z_1, z_2$  of the form

$$\partial^\mu(z_1 + e^{i\theta} \bar{z}_2) \partial_\mu(\bar{z}_1 + e^{-i\theta} z_2) + \partial^\mu(e^{-i\theta} z_2 - \bar{z}_1) \partial_\mu(e^{i\theta} \bar{z}_2 - z_1) + O\left(\frac{1}{t}\right)$$

---

<sup>6</sup> The degenerate holomorphic maps also contribute to the A-model [17], and lead to tree level Chern-Simons Lagrangian. These configurations will not be horizontal; however they will not affect our argument because the pair of boundaries ending on  $K$  in that case will cancel, as they correspond to open strings of infinitesimal width.

where  $\zeta = e^{i\theta}$  is the zero mode of  $\zeta$ , and the term  $O(1/t)$  comes from the integral over the non-zero modes. Notice that all the BRST non-invariance of this result is in the  $O(1/t)$  term. Dropping the  $O(1/t)$  terms, we end up with a path-integral in  $w_1, w_2$  which is the standard one for the  $A$ -model in flat space, with now a boundary condition which is a standard special Lagrangian brane in one less dimension, namely the curve  $F(z_1, z_2) = 0$  in  $\mathbf{C}^2$ . The path integral in  $z_1, z_2$  is then well-defined as a topological amplitude. We get a non-trivial contribution only for the discrete set of  $\theta$ 's for which there are BPS states with that central charge phase. Then the final integration over the  $\zeta$  zero-modes,  $\theta$ , may be replaced by a discrete sum.

In this way we see that, at least for large  $t$ , the BPS-counting path integral should be well-defined as a topological amplitude. As we have argued, it corresponds to a specific correlator in the CS theory on  $\Sigma \times S^1$ . The fact that the length of the  $S^1$  is taken to  $\infty$  is unimportant since CS is a strictly topological theory insensitive to metric deformations. This concludes the argument that this brane leads to sensible path-integral contributions for the  $A$ -model. It would be important to make this argument fully rigorous.

Even though we have a construction which captures the essence of what we need, it would have been more satisfactory if we had found a  $K$  which is special Lagrangian. It is not difficult to see that modifying the Kähler metric in the particular complex geometry we introduced will not lead to a Lagrangian  $K$ . It would be important to see if a relaxation of some of the other assumptions may lead to one. In fact the most natural choice for the restriction of the Kähler form to  $|\zeta| = 1$ , which has the required property as well as vanishing on  $K$  is  $e^{-i\theta} dz_1 \wedge dz_2 + c.c..$  This form is manifestly not closed. So we can make  $K$  Lagrangian at the expense of making the geometry non-Kählerian. This suggests that if we wish to have the 2-form vanish on  $K$ , we have to allow generalized complex and Kähler structures [22][23][24], which is natural if we have various kinds of fluxes turned on in the bulk. In fact a construction similar to what we need is already done in [25] (see also [26]) where it is shown how to construct a supersymmetric geometry using the twistor space with an  $SU(3)$  structure with a non-closed analog of Kähler form. The  $SU(3)$  structure uses a non-conventional complex structure on the twistor space which is not integrable and satisfies equations very much like what we need for our construction. Indeed this non-conventional complex structure for twistor space was already introduced in [27] in order to relate the problem of constructing harmonic maps from surfaces into four manifolds to holomorphic maps in the associated twistor space! It seems reasonable to expect that the most natural construction for our problem involves such a choice. We leave this for future studies.

## 6. Conclusion

We have seen that the BPS states of  $N = 2$  gauge theories in 4 dimensions are captured by open A-model topological string amplitudes. On the face of it, this seems in line with other appearances of BPS counting in topological strings [16]. However in those cases one typically computes only the electric BPS degeneracies. Here by a series of dualities we have been able to also realize counting of arbitrary electric and magnetic states in an associated A-model with higher bulk supersymmetry, whose supersymmetry is reduced by the presence of a Lagrangian brane. Thus for example for the A-model on toric geometry, the degeneracy of D2 and D4 branes get mapped to open A-model amplitudes realized on a twistorial space where the mirror of the toric geometry defines a brane.

We have offered a twistorial construction which captures what we need for our application. However, as we have noted, the condition of the A-brane being Lagrangian is not easy to satisfy in a Calabi-Yau background. It seems a better construction may require going to non-Kählerian geometries.

We have mainly focused on the case where the Seiberg-Witten curve is embedded in locally flat space. However there are more general cases (such as branes wrapping  $T^*C$ ) which are not of this type. It should be possible to extend the twistorial constructions to these cases as well [28].

As we have noted in the context of A-model topological string it is natural to consider an even more refined invariant, having to do with taking the multiplicity of M5 branes to be  $K > 1$ . The physical interpretation of the monodromy in this case would be interesting to explore. In particular the Wilson loop holonomies will be elements of  $U(K)$ .

It would be interesting to connect this work, with the approach taken in [9] in deriving the KS conjecture. In particular it would be interesting to translate the data for corrections to hyperkähler metric in the setup we have considered here.

This project is an offshoot of our work [28], in attempting to understand the meaning of KS monodromy. In analogy with the 2d case [2], one expects that the monodromy and its eigenvalues encode R-charge of chiral operators at conformal fixed points. The fact that monodromy captures R-charges in 2-dimensional theories was the highlight of those

results. It is natural to expect to have a similar story for 4d  $N = 2$  SCFT's. There are already encouraging signs in that direction [28].

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## References

- [1] E. Witten and D. I. Olive, “Supersymmetry Algebras That Include Topological Charges,” *Phys. Lett. B* **78**, 97 (1978).
- [2] S. Cecotti and C. Vafa, “On classification of N=2 supersymmetric theories,” *Commun. Math. Phys.* **158**, 569 (1993) [arXiv:hep-th/9211097].
- [3] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, “A New supersymmetric index,” *Nucl. Phys. B* **386**, 405 (1992) [arXiv:hep-th/9204102].
- [4] N. Seiberg and E. Witten, “Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory,” *Nucl. Phys. B* **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].
- [5] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. P. Warner, “Self-Dual Strings and N=2 Supersymmetric Field Theory,” *Nucl. Phys. B* **477**, 746 (1996) [arXiv:hep-th/9604034].
- [6] A. D. Shapere and C. Vafa, “BPS structure of Argyres-Douglas superconformal theories,” arXiv:hep-th/9910182 .
- [7] F. Denef and G. W. Moore, “Split states, entropy enigmas, holes and halos,” arXiv:hep-th/0702146.
- [8] M. Kontsevich and Y. Soibelman, “Stability Structures, Motivic Donaldson-Thomas Invariants and Cluster Transformations,” [arXiv:0811.2435].
- [9] D. Gaiotto, G. W. Moore and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” arXiv:0807.4723 [hep-th];  
D. Gaiotto, G. W. Moore and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” arXiv:0907.3987 [hep-th].
- [10] S. Katz, P. Mayr and C. Vafa, “Mirror symmetry and exact solution of 4D N = 2 gauge theories. I,” *Adv. Theor. Math. Phys.* **1**, 53 (1998) [arXiv:hep-th/9706110].
- [11] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” *Nucl. Phys. B* **577**, 419 (2000) [arXiv:hep-th/9912123].
- [12] A. Collinucci, P. Soler and A. M. Uranga, “Non-perturbative effects and wall-crossing from topological strings,” arXiv:0904.1133 [hep-th].
- [13] T. Dimofte and S. Gukov, “Refined, Motivic, and Quantum,” arXiv:0904.1420 [hep-th] ;  
T. Dimofte, S. Gukov and Y. Soibelman, to appear.
- [14] M. Marino and C. Vafa, “Framed knots at large N,” arXiv:hep-th/0108064.
- [15] E. Witten, “On The Structure of the Topological Phase of Two-Dimensional Gravity,” *Nucl. Phys. B* **340**, 281 (1990).
- [16] R. Gopakumar and C. Vafa, “M-theory and topological strings. I,” arXiv:hep-th/9809187 ;

- R. Gopakumar and C. Vafa, “M-theory and topological strings. II,” arXiv:hep-th/9812127.
- [17] E. Witten, “Chern-Simons Gauge Theory As A String Theory,” *Prog. Math.* **133**, 637 (1995) [arXiv:hep-th/9207094].
  - [18] H. Ooguri and C. Vafa, “Two-Dimensional Black Hole and Singularities of CY Manifolds,” *Nucl. Phys. B* **463**, 55 (1996) [arXiv:hep-th/9511164].
  - [19] E. Witten, “Solutions of four-dimensional field theories via M-theory,” *Nucl. Phys. B* **500**, 3 (1997) [arXiv:hep-th/9703166].
  - [20] D. Gaiotto, “N=2 dualities,” arXiv:0904.2715 [hep-th].
  - [21] E. Witten, “Quantum field theory and the Jones polynomial,” *Commun. Math. Phys.* **121**, 351 (1989).
  - [22] C. M. Hull, U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, “Generalized Kahler geometry and gerbes,” arXiv:0811.3615 [hep-th].
  - [23] A. Kapustin and Y. Li, “Topological sigma-models with H-flux and twisted generalized complex manifolds,” arXiv:hep-th/0407249.
  - [24] V. Pestun, “Topological strings in generalized complex space,” *Adv. Theor. Math. Phys.* **11**, 399 (2007) [arXiv:hep-th/0603145].
  - [25] A. Tomasiello, “New string vacua from twistor spaces,” *Phys. Rev. D* **78**, 046007 (2008) [arXiv:0712.1396 [hep-th]].
  - [26] A. D. Popov, “Hermitian-Yang-Mills equations and pseudo-holomorphic bundles on nearly arXiv:0907.0106 [hep-th].
  - [27] J. Eells and S. Salamon, “Twistorial construction of harmonic maps of surfaces into four-manifolds,” *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **4** no. 4, 12 (1985) 589.
  - [28] S. Cecotti, A. Neitzke and C. Vafa, work in progress